REMARKS/ARGUMENTS

Reconsideration of this application is respectfully requested.

In response to the Examiner's objection to claim 13, the Examiner is respectfully referred to Figure 11 and the corresponding text of applicant's specification at page 11. In this exemplary embodiment, the Hilbert transform filter corresponds to the phase shifter, the complex modulator 6 upconverts the complex spread spectrum signal and the mixer 9 is a data modulator which is arranged in conjunction with the low pass filter 10 to provide a demodulated output. Accordingly, in accordance with the applicant's chosen lexicography, it is believed that claim 13 is correctly stated.

In response to the rejection of claim 1 under 35 U.S.C. §112, claim 1 has been amended so as to avoid a possible lack of antecedent basis for the input data signal. Claim 7 has also been slightly amended so as to remove the recited input signal from the preamble since it is more properly already recited in the body of the claim in a manner that does not require antecedent basis. A typographical error has been corrected at claim 13. Although the body of both claims 11 and 13 already make it clear that the signal at issue is a spread spectrum signal, the preamble to these claims has also been amended to this same effect.

Since the presented amendments are only minor formality-related amendments, they are not believed to present any new issues and are believed to be properly enterable under the provisions of 37 C.F.R. §1.116.

The rejection of claims 1, 2, 4-11 and 13 under 35 U.S.C. §102 as allegedly anticipated by the newly cited Kumar '894 reference is respectfully traversed.

All of applicant's claims are directed towards method or apparatus for generating or decoding a <u>spread spectrum</u> signal. The term "spread spectrum" is a well understood term of art which requires, <u>inter alia</u>, that the signaling bandwidth must be much wider than the information bandwidth. For example, the carrier may be "spread" in its spectrum by modulation with a "spreading function" -- <u>in addition</u> to modulating the spread spectrum signal with data (i.e., information). For the Examiner's convenience, a tutorial on spread spectrum (accessed via the internet and a Google search) is attached.

Contrary to the Examiner's assertion, Kumar '894 does <u>not</u> relate at all to a "spread spectrum" system. Instead, Kumar '894 relates to a conventional narrow band system. In particular, Kumar teaches various techniques for creating a composite of additional information bearing signal envelopes that can be transmitted along with a conventional amplitude modulated band signal envelope (i.e., such that the entire signaling spectrum fits within the narrow band frequency "mask" allocated by the FCC to a conventional AM commercial radio station).

While some Kumar embodiments involve converting the AM radio analog audio information into a single sideband signal using a Hilbert Transform filter (i.e., to effect a 90° phase shift for each frequency component of the processed audio signal), Kumar's use of the Hilbert Transform filter is not related to generation of any spread spectrum signal.

Some brief explanations of the Hilbert Transform (also acquired via the internet and Google are also attached for the Examiner's general information.

Taking the Examiner's more detailed assertions in turn, the Examiner first alleges that Kumar teaches generation of a complex spreading signal in: the Abstract, at column 4, lines 9-14, 28-33 and at column 10, lines 10-43. However, that is not the case:

Kumar Abstract

The Abstract does not mention a spreading signal of any kind -- let alone a complex-valued spreading signal. Indeed, to the extent that the Abstract deals at all with the spectrum occupied by the Kumar signals, the Abstract explicitly teaches that the composite RF signal includes a digital signal "whose spectrum is substantially confined in one inner sideband" of the standard FCC frequency allocation to an AM broadcast station. The Abstract also makes it clear that the analog monophonic component of the composite signaling may be received by any conventional AM-band audio receiver -- even if the audio portion of the composite signal has been transformed into a single sideband signal. In other words, the Abstract makes it clear that the Kumar composite signals are all well contained within a conventional narrow band signaling channel (i.e., the usual narrow band "mask" granted to a given AM commercial station by the FCC).

Kumar's Column 4, lines 9-14

This portion of Kumar merely describes the fact that there is no available extra space (i.e., bandwidth) at the inner sidebands of a conventional analog AM stereo broadcasting signal -- and thus any additional digital signaling that might be

possible to composite with such signaling would have to go in the outer sideband areas of the FCC frequency allocation:

"The implementation of analog AM stereo broadcasting according to the Kahn/Hazeltine system precludes the generation of a supplemental digital signal in the upper inner or lower inner sidebands because all available signaling dimensions are utilized, and there is substantially no unoccupied band width".

Kumar's Column 4, lines 28-33

This section of Kumar is merely describing yet another prior art situation as depicted in Figure 6 where certain forms of quadrature amplitude modulation within the FCC allocated frequency "mask" associated with a given station actually extends somewhat beyond the inner sideband areas:

"The presence of the amplitude-limiter results in a non-linear method of modulation. The resulting signals bandwidth as determined by the spectrum occupancy of exemplary C-QUAM upper sideband 32 and C-QUAM lower sideband 34 in Fig. 6 is greater than when a substantially linear method of modulation such as conventional amplitude modulation (AM) or quadrature amplitude modulation (QAM), is implemented.

Kumar's Column 10, lines 10-43

This paragraph of text is too lengthy to quote in full but it simply describes Kumar's various proposals for compressing the analog audio information in the AM band (e.g., by converting it to a single sideband signal) so as to thus provide a little more room within the FCC frequency allocation "mask" for additional composite signaling to be added according to Kumar's desires.

None of this has anything whatever to do with the generation of a spreading signal for use in a spread spectrum environment -- let alone a complex spreading signal as required by applicant's claim 1.

The Examiner next asserts that Kumar teaches phase shifting of a complex spreading signal in accordance with a Hilbert transform to produce a phase shifted

complex spreading signal at column 29, lines 13-50. However, this passage of Kumar merely relates to the Kumar embodiment of Figures 11/15 where a Hilbert transform filter 205 is used to effect a 90° phase shift of all frequency components of the input analog audio signal. That is, as shown in Figure 11, the analog audio signal has already been converted to digital format at 57 and therefore the phase shifting involved in creating a single sideband version of this information utilizes the digital Hilbert transform filter 205 shown in Figure 15. However, the signal being phase shifted by the Hilbert transform filter 205 is the monophonic <u>audio</u> input signal 59 -- <u>not</u> a complex spreading signal. Furthermore, the output of the Hilbert Transform filter is not a phase shifted complex spreading signal. Instead, it is a phase shifted version of the audio input that is used at mixer 209 for generating one component of what will ultimately become a single sideband version of the analog audio information.

Next, the Examiner alleges that Kumar teaches upconverting the complex spreading signal and the phase shifted complex spreading signal to a higher frequency to produce a single sideband spread spectrum signal at Figure 11, column 21, lines 44-59. However, Figure 11 actually includes a specific spectral diagram demonstrating exactly how the composite signaling envelopes A, D1 and D2 are fitted within the FCC allocated frequency "mask" for a conventional narrow AM band commercial station. The RF upconvertor 99 is merely upconverting the composite single sideband signal components A, D1 and D2 onto a desired center frequency. There is no upconversion of any complex

spreading signal nor of any phase shifted complex spreading signal. Nor does the circuitry of Figure 11 ever produce a single sideband spread spectrum signal.

The text at column 21, lines 44-59 is to the same effect. There is no mention anywhere in this text of any complex spreading signal, phase shifted complex spreading signal or any single sideband spread spectrum signal.

Without belaboring the point, the remainder of the Examiner's assertions with respect to claim 1 are also clearly erroneous for similar and analogous reasons.

Independent claim 7 is directed towards apparatus for transmitting a single sideband spread spectrum signal and also requires, inter alia, a complex spreading signal generator, a phase shifter coupled to receive the complex spreading signal and a phase shift in accordance with a Hilbert Transform, etc. For reasons analogous to those already discussed, there is no possible teaching or suggestion of the apparatus of claim 7 anywhere in Kumar.

Independent claim 11 recites a method of decoding a single sideband spread spectrum signal which requires, inter alia, upconverting a complex spreading signal to a higher frequency and then using that upconverted complex spreading signal to demodulate a received signal. As already mentioned, there is nothing in Kumar about spread spectrum technology whatsoever -- let alone any upconversion of a complex spreading signal.

Independent claim 13 is directed to an apparatus for decoding a transmitted spread spectrum signal and requires, inter alia, a complex spreading signal generator, a phase shifter connected to phase shift the output of the signal generator, a complex modulator which upconverts the complex spreading signal, etc. Once again, there is no possible teaching or suggestion of anything with respect to spread spectrum technology anywhere in the Kumar patent.

Given the very fundamental deficiencies of Kumar with respect to independent claims as already noted, it is not believed necessary at this time to detail the additional deficiencies of this essentially irrelevant reference with respect to applicant's dependent claims.

Accordingly, this entire application is now believed to be in allowable condition and a formal Notice to that effect is respectfully solicited.

Respectfully submitted,

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Spread Spectrum Scen

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The ABCs of Spread Spectrum -- *A Tutorial*

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Introduction to Spread Spectrum

by Randy Roberts, Director of RF/Spread Spectrum Consulting (Retired)

Over the last eight or nine years a new commercial marketplace has been emerging. Called spre spectrum, this field covers the art of secure digital communications that is now being exploited f commercial and industrial purposes. In the next several years hardly anyone will escape being involved, in some way, with spread spectrum communications. Applications for commercial spr spectrum range from "wireless" LAN's (computer to computer local area networks), to integrate bar code scanner/palmtop computer/radio modem devices for warehousing, to digital dispatch, t digital cellular telephone communications, to "information society" city/area/state or country wi networks for passing faxes, computer data, email, or multimedia data.

The *IEEE Spectrum* of August, 1990 contained an article entitled *Spread Spectrum Goes Commercial,* by Donald L. Schilling of City College of New York, Raymond L. Pickholtz of George Washington University, and Laurence B. Milstein of UC San Diego. This article summarized the coming of commercial spread spectrum:

> "Spread-spectrum radio communications, long a favorite technology of the military because it resists jamming and is hard for an enemy to intercept, is now on the verge of potentially explosive commercial development. The reason: spread-spectrum signals, which are distributed over a wide range of frequencies and then collected onto their original frequency at the receiver, are so inconspicuous as to be 'transparent.' Just as they are unlikely to be intercepted by a military opponent, so are they unlikely to interfere with other signals intended for business and consumer users -- even ones transmitted on the same frequencies. Such an advantage opens up crowded frequency spectra to vastly expanded use.
>
> "A case in point is a two-year demonstration project the Federal Communications Commission (FCC) authorized in May (1990) for Houston, Texas, and Orlando, Fla. In both places, a new spread spectrum personal communications network (PCN) will share the 1.85-1.9-gigahertz band with local electric and gas utilities. The FCC licensee, Millicom Inc., a New York City-based cellular telephone company, expects to enlist 45000 subscribers.
>
> "The demonstration is intended to show that spread-spectrum users can share a frequency band with conventional microwave radio users--without one group interfering with the other -- thereby increasing the efficiency with which that band is used. . . . "

How Spread Spectrum Works

Spread Spectrum uses wide band, noise-like signals. Because Spread Spectrum signals are noise like, they are hard to detect. Spread Spectrum signals are also hard to Intercept or demodulate. Further, Spread Spectrum signals are harder to jam (interfere with) than narrowband signals. Th Low Probability of Intercept (LPI) and anti-jam (AJ) features are why the military has used Spre Spectrum for so many years. Spread signals are intentionally made to be much wider band than t information they are carrying to make them more noise-like.

Spread Spectrum signals use fast codes that run many times the information bandwidth or data r These special "Spreading" codes are called "Pseudo Random" or "Pseudo Noise" codes. They ar called "Pseudo" because they are not real gaussian noise.

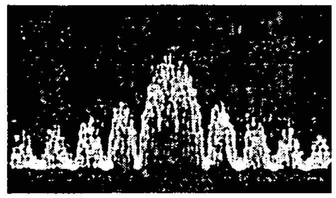
Spread Spectrum transmitters use similar transmit power levels to narrow band transmitters. Because Spread Spectrum signals are so wide, they transmit at a much lower spectral power den measured in Watts per Hertz, than narrowband transmitters. This lower transmitted power densit characteristic gives spread signals a big plus. Spread and narrow band signals can occupy the sa band, with little or no interference. This capability is the main reason for all the interest in Sprea Spectrum today.

More Details on Spread Spectrum

Over the last 50 years, a class of modulation techniques usually called "Spread Spectrum," has b developed. This group of modulation techniques is characterized by its wide frequency spectra. ' modulated output signals occupy a much greater bandwidth than the signal's baseband informati bandwidth. To qualify as a spread spectrum signal, two criteria should be met:

1. The transmitted signal bandwidth is much greater than the information bandwidth.

2. Some function other than the information being transmitted is employed to determine the resultant transmitted bandwidth.



A Spectrum Analyzer Photo of a Direct Sequence (DS) Spread Spectrum signal.

Most commercial part 15.247 spread spectrum systems transmit an RF signal bandwidth as wide 20 to 254 times the bandwidth of the information being sent. Some spread spectrum systems hav employed RF bandwidths 1000 times their information bandwidth. Common spread spectrum systems are of the "direct sequence" or "frequency hopping" type, or else some combination of t two types (called a "hybrid").

A Spectrum Analyzer Photo of a Frequency Hop (FH) Spread Spectrum signal.

There are also "Time Hopped" and "Chirp" systems in existence. Time hopped spread spectrum systems have found no commercial application to date. However, the arrival of cheap random access memory (RAM) and fast micro-controller chips make time hopping a viable alternative spread spectrum technique for the future. "Chirp" signals are often employed in radar systems ar only rarely used in commercial spread spectrum systems.

Direct sequence systems -- Direct sequence spread spectrum systems are so called because they employ a high speed code sequence, along with the basic information being sent, to modulate th RF carrier. The high speed code sequence is used directly to modulate the carrier, thereby direct setting the transmitted RF bandwidth. Binary code sequences as short as 11 bits or as long as [2^ (89) - 1] have been employed for this purpose, at code rates from under a bit per second to sever hundred megabits per second.

The result of modulating an RF carrier with such a code sequence is to produce a signal centerec the carrier frequency, direct sequence modulated spread spectrum with a (sin x/x)2 frequency spectrum. The main lobe of this spectrum has a bandwidth twice the clock rate of the modulatin code, from null to null. The sidelobes have a null to null bandwidth equal to the code's clock rate Figure 1 illustrates the most common type of direct sequence modulated spread spectrum signal.

Direct sequence spectra vary somewhat in spectral shape depending upon the actual carrier and (modulation used. The signal illustrated is that for a binary phase shift keyed (BPSK) signal, whi is the most common modulation signal type used in direct sequence systems.

Frequency hopping systems -- The wideband frequency spectrum desired is generated in a differ manner in a frequency hopping system. It does just what its name implies. That is, it "hops" fron frequency to frequency over a wide band. The specific order in which frequencies are occupied i function of a code sequence, and the rate of hopping from one frequency to another is a function the information rate. The transmitted spectrum of a frequency hopping signal is quite different fi that of a direct sequence system. Instead of a [(sin x)/x]^2-shaped envelope, the frequency hopp output is flat over the band of frequencies used. Figure 2 shows an output spectrum of a frequen hopping system. The bandwidth of a frequency hopping signal is simply w times the number of frequency slots available, where w is the bandwidth of each hop channel.

"Inside" Spread Spectrum

This section is intended to gently introduce the reader to the more intricate aspects of the rapidly growing world of spread spectrum, wireless local and wide area networks, as well as introduce tl evolution (some may call it explosion) in new communications technologies such as PCN/PCS. will also try to thoroughly define new terms and concepts the first time we use them.

As an introduction, a little history lesson and a few definitions seem to be in order. Spread Spect (SS) dates back to World War II. A German lady scientist was granted a patent on a simple frequency hopping CW system. The allies also experimented with spread spectrum in World Wa These early research and development efforts tried to provide countermeasures for radar, naviga beacons and communications. The U. S. Military has used SS signals over satellites for at least 2 years. An old, but faithful, highly capable design like the Magnavox USC-28 modem is an exam of this kind of equipment. Housed in two or three six foot racks, it had selectable data rates from few hundred bits per second to about 64 kBits per second. It transmitted a spread bandwidth of 6 MHZ. Many newer commercial satellite systems are now converting to SS to increase channel capacity and reduce costs.

Over the last twenty years, many spread spectrum signals have appeared on the air. The easiest v to characterize these modulations is by their frequency spectra. These SS signals occupy a much greater bandwidth than needed by the information bandwidth of the transmitted data. To rate bei called an SS signal, two technicalities must be met:

- The signal bandwidth must be much wider than the information bandwidth.
- Some code or pattern, other than the data to be transmitted, determines the actual on-the-a transmit bandwidth.

In today's commercial spread spectrum systems, bandwidths of 10 to 100 times the information rates are used. Military systems have used spectrum widths from 1000 to 1 million times the information bandwidth. There are two very common spread spectrum modulations: frequency hopping and direct sequence. At least two other types of spreading modulations have been used: time hopping and chirp.

What Exactly is Spread Spectrum?

One way to look at spread spectrum is that it trades a wider signal bandwidth for better signal to noise ratio. Frequency hop and direct sequence are well-known techniques today. The following paragraphs will describe each of these common techniques in a little more detail and show that pseudo noise code techniques provide the common thread through all spread spectrum types.

Frequency hopping is the easiest spread spectrum modulation to use. Any radio with a digitally controlled frequency synthesizer can, theoretically, be converted to a frequency hopping radio. 1 conversion requires the addition of a pseudo noise (PN) code generator to select the frequencies transmission or reception. Most hopping systems use uniform frequency hopping over a band of frequencies. This is not absolutely necessary, if both the transmitter and receiver of the system k in advance what frequencies are to be skipped. Thus a frequency hopper in two meters, could be made that skipped over commonly used repeater frequency pairs. A frequency hopped system cε use analog or digital carrier modulation and can be designed using conventional narrow band ra(techniques. De-hopping in the receiver is done by a synchronized pseudo noise code generator tl drives the receiver's local oscillator frequency synthesizer.

The most practical, all digital version of SS is direct sequence. A direct sequence system uses a locally generated pseudo noise code to encode digital data to be transmitted. The local code runs much higher rate than the data rate. Data for transmission is simply logically modulo-2 added (a EXOR operation) with the faster pseudo noise code. The composite pseudo noise and data can b passed through a data scrambler to randomize the output spectrum (and thereby remove discrete spectral lines). A direct sequence modulator is then used to double sideband suppressed carrier modulate the carrier frequency to be transmitted. The resultant DSB suppressed carrier AM modulation can also be thought of as binary phase shift keying (BPSK). Carrier modulation othe than BPSK is possible with direct sequence. However, binary phase shift keying is the simplest a most often used SS modulation technique.

An SS receiver uses a locally generated replica pseudo noise code and a receiver correlator to separate only the desired coded information from all possible signals. A SS correlator can be thought of as a very special matched filter -- it responds only to signals that are encoded with a pseudo noise code that matches its own code. Thus, an SS correlator can be "tuned" to different codes simply by changing its local code. This correlator does not respond to man made, natural (artificial noise or interference. It responds only to SS signals with identical matched signal characteristics and encoded with the identical pseudo noise code.

What Spread Spectrum Does

The use of these special pseudo noise codes in spread spectrum (SS) communications makes sig appear wide band and noise-like. It is this very characteristic that makes SS signals possess the quality of Low Probability of Intercept. SS signals are hard to detect on narrow band equipment because the signal's energy is spread over a bandwidth of maybe 100 times the information bandwidth.

The spread of energy over a wide band, or lower spectral power density, makes SS signals less likely to interfere with narrowband communications. Narrow band communications, conversely, cause little to no interference to SS systems because the correlation receiver effectively integrate over a very wide bandwidth to recover an SS signal. The correlator then "spreads" out a narrow band interferer over the receiver's total detection bandwidth. Since the total integrated signal der or SNR at the correlator's input determines whether there will be interference or not. All SS systı have a threshold or tolerance level of interference beyond which useful communication ceases. 1

tolerance or threshold is related to the SS processing gain. Processing gain is essentially the ratic the RF bandwidth to the information bandwidth.

A typical commercial direct sequence radio, might have a processing gain of from 11 to 16 dB, depending on data rate. It can tolerate total jammer power levels of from 0 to 5 dB stronger than desired signal. Yes, the system can work at negative SNR in the RF bandwidth. Because of the processing gain of the receiver's correlator, the system functions at positive SNR on the baseban data.

Besides being hard to intercept and jam, spread spectrum signals are hard to exploit or spoof. Si exploitation is the ability of an enemy (or a non-network member) to listen in to a network and u information from the network without being a valid network member or participant. Spoofing is act of falsely or maliciously introducing misleading or false traffic or messages to a network. SS signals also are naturally more secure than narrowband radio communications. Thus SS signals c be made to have any degree of message privacy that is desired. Messages can also, be cryptographically encoded to any level of secrecy desired. The very nature of SS allows military intelligence levels of privacy and security to be had with minimal complexity. While these characteristics may not be very important to everyday business and LAN (local area network) ne these features are important to understand.

Some Spread Spectrum Terms Defined

Spread spectrum technology seems to present an alphabet soup to most newcomers. We define s of the more commonly used terms in this field in the following text box. For a complete glossary see our complete Glossary.

A Brief Spread Spectrum Glossary

For more definitions of spread spectrum terms, please visit our Technical Glossary.

- **AJ:** Anti-Jam, designed to resist interference or jamming.
- **BPSK:** Binary Phase Shift Keying -- Digital DSB suppressed carrier modulation.
- **CDMA:** Code Division Multiple Access -- a way to increase channel capacity.
- **CHIP:** The time it takes to transmit a bit or single symbol of a PN code.
- **CODE:** A digital bit stream with noise-like characteristics.
- **CORRELATOR:** The SS receiver compponent that demodulates a Spread Spectrum signal.
- **DE-SPREADING:** The process used by a correlator to recover narrowband information from a spread spectrum signal.
- **WIRELESS LAN:** Wireless Local Area Network - a 1,000-foot or less range computer-to-computer data communications network.
- **PCN:** Personal Communication Network. PCNs are usually short range (hundreds of feet to 1 mile or so) and involve cellular radio type architecture. Services include digital voice, FAX, mobile data and national/international data communications.

- **PCS:** Personal Communication System. PCSs are usually associated with cordless telephone type devices. Service is typically digital voice only.
- **PN:** Pseudo Noise - a digital signal with noise-like properties.
- **RF:** Radio Frequency - generally a frequency from around 50 kHz to around 3 GHz. RF is usually referred to whenever a signal is radiated through the air.
- **SS:** Spread Spectrum, a wideband modulation which imparts noise-like characteristics to an RF signal.
- **WIRELESS UAN:** Wireless Universe Area Network - a collection of wireless MANs or WANs that link together an entire nation or the world. UANs use very small aperture (VSAT) earth station gateway technology.

Conclusion

Our world is rapidly changing -- computers have gone from mainframes to palmtops. Radio communications has gone from lunchbox sized (or trunk mounted/remote handset car phone) to cigarette-pack-sized micro-cellular telephone technology. The technical challenges of this progr are significant. The new opportunities created by this new technology are also significant. We've talked here about some of the very basic principles in spread spectrum and talked about evolving career opportunities -- isn't it time somebody did something about moving forward in the new millennium?

About the Author:

Randy Roberts has over 30 years experience in communications, electronics and spread spectru system design. He graduated with a BSEE in 1970 from UC Irvine. For many years prior to his retirement he operated RF/Spread Spectrum Consulting, an independent product development, publishing, strategic planning and training company. He is the founder and former publisher of Spread Spectrum Scene Online.

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Revised January 23, 2006

mathworld

Hilbert Transform

The Hilbert transform (and its inverse) are the integral transform

$$g(y) = \mathcal{H}[f(x)] = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-y} \qquad (1$$

$$f(x) = \mathcal{H}^{-1}[g(y)] = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{g(y)\,dy}{y-x}, \qquad (2$$

where the Cauchy principal value is taken in each of the integrals. The Hilbert transform is an improper integral.

In the following table, $\Pi(x)$ is the rectangle function, $\operatorname{sinc}(x)$ is the sinc function, $\delta(x)$ is the delta function, $\mathrm{II}(x)$ and ${}^{\mathrm{I}}\mathrm{I}(x)$ are impulse symbols, and ${}_1F_1(a;b;x)$ is a confluent hypergeometric function of the first kind.

$f(x)$	$g(y)$
$\sin x$	$\cos y$
$\cos x$	$-\sin y$
$\frac{\sin x}{x}$	$\frac{\cos y - 1}{y}$
$\Pi(x)$	$\frac{1}{\pi} \ln \left\lvert \frac{y-\frac{1}{2}}{y+\frac{1}{2}} \right\rvert$
$\frac{1}{1+x^2}$	$-\frac{y}{1+y^2}$
$\operatorname{sinc}'(x)$	$\frac{1-\cos y - y \sin y}{y^2}$
$\delta(x)$	$-\frac{1}{\pi y}$
$\mathrm{II}(x)$	$\frac{x}{\pi\left(\frac{1}{4}-y^2\right)}$
${}^{\mathrm{I}}\mathrm{I}(x)$	$-\frac{1}{2\pi\left(\frac{1}{4}-y^2\right)}$
e^{-x^2}	$-e^{-y^2} \operatorname{erfi}(y)$

SEE ALSO: Abel Transform, Fourier Transform, Improper Integral, Integral Transform, Titchmarsh Theorem, Wiener-Lee Transform. [Pages Linking Here]

REFERENCES:

Bracewell, R. "The Hilbert Transform." *The Fourier Transform and Its Applications, 3rd ed.* New York: McGraw-Hill, pp. 267-272, 1999.

Papoulis, A. "Hilbert Transforms." *The Fourier Integral and Its Applications.* New York: McGraw Hill, pp. 198-201, 1962.

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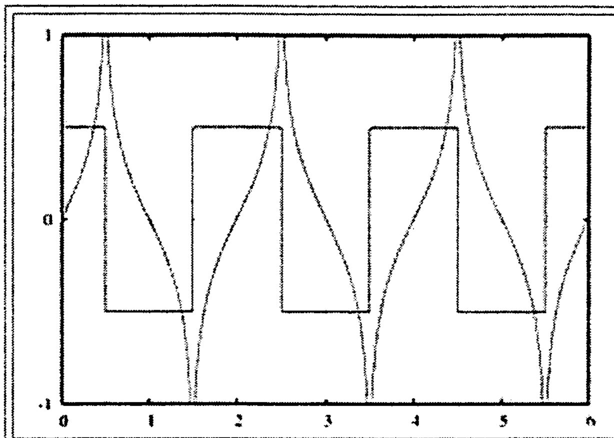
Hilbert transform

From Wikipedia, the free encyclopedia

In mathematics and in signal processing, the **Hilbert transform**, here denoted $\mathcal{H}$, of a real-valued function, $s(t)$, is obtained by convolving signal $s(t)$ with $1 / (\pi t)$ to obtain $\widehat{s}(t)$. Therefore, the Hilbert transform $\widehat{s}(t)$ can be interpreted as the output of a linear time invariant system with input $s(t)$, and a system impulse response given as $1 / (\pi t)$. It is a useful mathematical tool to describe the complex envelope of a real-valued carrier modulated signal in communication theory (see below for more on applications). The precise definition is as follows:



The Hilbert transform, in red, of a square wave, in blue

$$\widehat{s}(t) = \mathcal{H}\{s\}(t) = (h * s)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{s(\tau)}{t-\tau}\,d\tau.$$

where

$$h(t) = \frac{1}{\pi t}$$

and considering the integral as a Cauchy principal value (which avoids the singularity at $\tau = t$).

It follows that the Hilbert transform has a frequency response given by the Fourier transform:

$$\begin{aligned} H(\omega) = \mathcal{F}\{h\}(\omega) &= -j \cdot \operatorname{sgn}(\omega), \text{ where } j \text{ (aka } i \text{) is the imaginary unit} \\ &= +j, \text{ for } \omega < 0 \\ &= -j, \text{ for } \omega > 0 \end{aligned}$$

And since:

$$\mathcal{F}\{\widehat{s}\}(\omega) = H(\omega) \cdot \mathcal{F}\{s\}(\omega),$$

the Hilbert transform has the effect of shifting the negative frequency components of $s(t)$ by +90° and the positive frequencies components by −90°.

We also note that $H^2(\omega) = -1$. So multiplying the above equation by $-H(\omega)$ gives

$$\mathcal{F}\{s\}(\omega) = -H(\omega) \cdot \mathcal{F}\{\widehat{s}\}(\omega)$$

from which the **inverse Hilbert transform** is apparent:

$$s(t) = -(h * \widehat{s})(t) = -\mathcal{H}\{\widehat{s}\}(t).$$

Contents

Hilbert transform examples

Notice: Some authors, e.g., Bracewell, use our $-\mathcal{H}$ as their definition of the forward transform. A consequence is that the right column of this table would be negated.

Signal $s(t)$	**Hilbert transform** $\mathcal{H}\{s\}(t)$
$\sin(t)$	$-\cos(t)$
$\cos(t)$	$\sin(t)$
$\frac{1}{t^2+1}$	$\frac{t}{t^2+1}$
$\frac{\sin(t)}{t}$ **Sinc function**	$\frac{1-\cos(t)}{t}$
$\sqcap(t)$ **Rectangular function**	$\frac{1}{\pi}\ln\left\|\frac{t+\frac{1}{2}}{t-\frac{1}{2}}\right\|$
$\delta(t)$ **Delta function**	$\frac{1}{\pi t}$

Narrowband model

Many signals can be accurately modeled as the product of a bandlimited "message" waveform, $s_m(t)$, and a sinusoidal "carrier":

$$s(t) = s_m(t) \cdot \cos(\omega t + \varphi)$$

When $s_m(t)$ has no frequency content above the carrier frequency, $\frac{\omega}{2\pi}$ Hz, then:

$$\widehat{s}(t) = s_m(t) \cdot \sin(\omega t + \varphi)$$

So, the Hilbert transform may be as simple as a circuit that produces a 90° phase shift at the carrier frequency. Furthermore:

$$(\omega t + \varphi)_{\mathrm{mod}\ 2\pi} = \arctan\left(\frac{\widehat{s}(t)}{s(t)}\right)$$

from which one can reconstruct the carrier waveform. Then the message can be extracted from $s(t)$ by coherent demodulation.

Analytic representation

The analytic representation of a signal is defined in terms of the Hilbert transform:

$$s_a(t) = s(t) + j \cdot \widehat{s}(t)$$

E.g., for the narrowband model [above], the analytic representation is:

$$\begin{aligned} s_a(t) &= s_m(t) \cdot \cos(\omega t + \varphi) + j \cdot s_m(t) \cdot \sin(\omega t + \varphi) \\ &= s_m(t) \cdot \left[\cos(\omega t + \varphi) + j \cdot \sin(\omega t + \varphi)\right] \\ &= s_m(t) \cdot e^{j(\omega t + \varphi)} \quad \text{(by Euler's formula)} \end{aligned}$$

This complex heterodyne operation shifts all the frequency components of $s_m(t)$ above 0 Hz. In that case, the imaginary part of the result is a Hilbert transform of the real part. So that is an indirect way to produce Hilbert transforms.

Practical considerations

The function *h* with $h(t) = 1/(\pi t)$ is a non-causal filter and therefore cannot be implemented as is, if *s* is a time-dependent signal. If *s* is a function of a non-temporal variable, e.g., spatial, the non-causality might not be a problem. The filter is also of infinite support which may be a problem in certain applications. Another issue relates to what happens with the zero frequency (DC), which can be avoided by assuring that *s* does not contain a DC-component.

A practical implementation in many cases implies that a finite support filter, which in addition is made causal by means of a suitable delay, is used to approximate the computation. The approximation may also imply that only a specific frequency range is subject to the characteristic phase shift related to the Hilbert transform. See also quadrature filter.

Discrete Hilbert transform

If the signal $s(t)$ is bandlimited, then $\widehat{s}(t)$ is bandlimited in the same way. Consequently, both these signals

can be sampled according to the sampling theorem, resulting in the discrete signals $s[n]$ and $\widehat{s}[n]$. The relation between the two discrete signals is then given by the convolution:

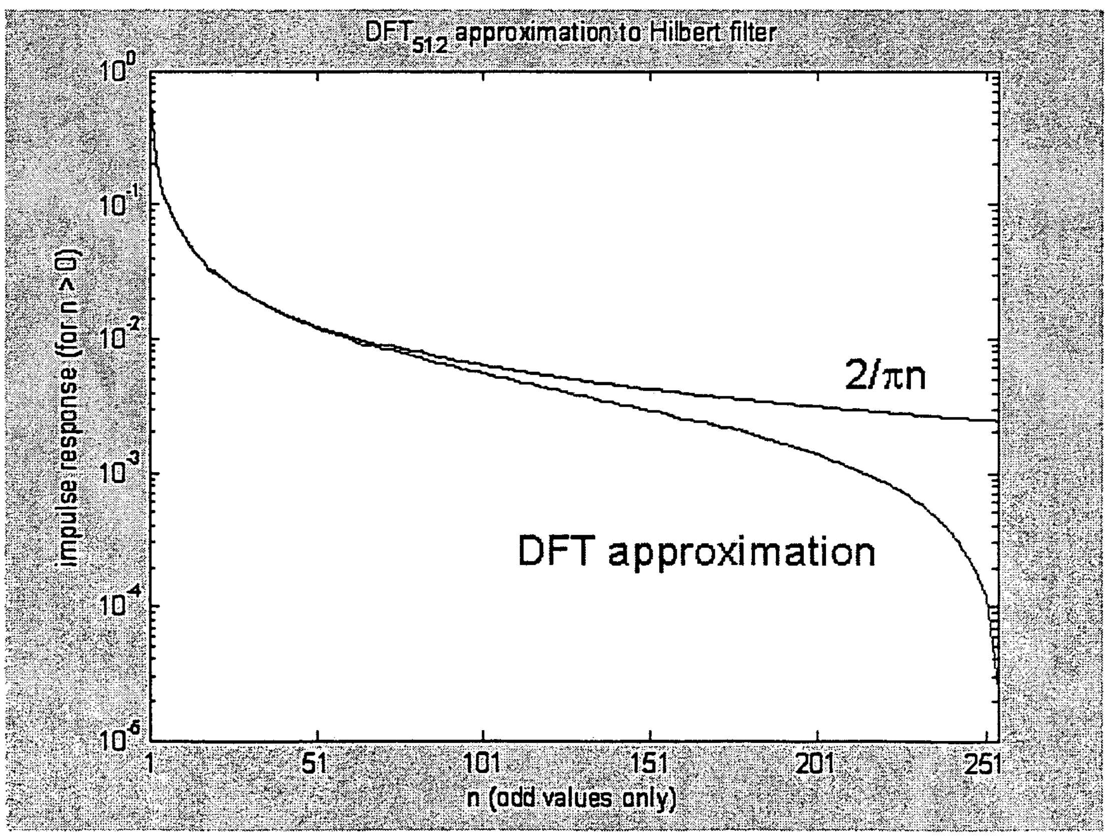
$$\widehat{s}[n] = h[n] * s[n]$$

where

$$h[n] = \begin{cases} 0, & \text{for } n \text{ even,} \\ \frac{2}{\pi n} & \text{for } n \text{ odd} \end{cases}$$

which is non-causal and has infinite duration. In practice, a shortened and time-shifted approximation is used. The usual filter design tradeoffs apply (e.g. filter-order and latency vs. frequency-response). Also notice, that $h[n]$ is not just a sampled version of the *Hilbert filter* $h(t)$, defined above. Rather it is a sequence with this discrete-time Fourier transform:

$$H(e^{j\omega}) = \begin{cases} +j, & -\pi \leq \omega < 0 \\ -j, & 0 \leq \omega < \pi \end{cases}$$



We note that a sequence similar to $h[n]$ can be generated by sampling $H(e^{j\omega})$ and computing the inverse discrete Fourier transform. The larger the transform (i.e. more samples per 2π radians), the better the agreement (for a given value of the abscissa, n). The figure shows the comparison for a 512-point transform. (Due to odd-symmetry, only half the sequence is actually plotted.)

But that is not the actual point, because it is easier and more accurate to generate $h[n]$ directly from the formula. The point is that many applications choose to avoid the convolution by doing the equivalent frequency-domain

operation: simple multiplication of the signal transform with the samples of $H(e^{j\omega})$, made even easier by the fact that the real and imaginary components are 0 and ±1 respectively. After transforming back to the time-domain, those applications have indirectly generated (and convolved with) not $h[n]$, but the DFT approximation to it. *Frequency-domain filtering* is often called *fast convolution*.

Fast convolution filters the signal piecewise, and the outputs are subsequently pieced back together. An important issue to understand about that approach is circular convolution. It is a type of distortion that can be avoided by overlapping the segments and choosing a segment size usually several times larger than the duration of the filter impulse response. But when the DFT approximation is used instead of a designed filter, the impulse response duration equals the segment length, and circular convolution cannot be avoided completely. However, it can be made arbitrarily small with an appropriate choice of the segment length.

See also

- Analytic signal
- Single-sideband signal
- Harmonic conjugate
- Kramers-Kronig relations

References

- Bracewell, R; The Fourier Transform and Its Applications, 2nd ed, 1986, McGraw-Hill.
- Carlson, A. Bruce; Crilly, Paul B.; & Rutledge, Janet C. (2002). Communication Systems (4th ed.).

External links

- Hilbert transform (http://www.ee.byu.edu/ee/class/ee444/ComBook/ComBook/node19.html)
- another exposition (http://cyvision.if.sc.usp.br/visao/courses/hilbert/hilbert.htm) — Hilbert transform properties (http://cyvision.if.sc.usp.br/visao/courses/hilbert/hproprie.htm)
- Mathworld Hilbert transform (http://mathworld.wolfram.com/HilbertTransform.html) — Contains a table of transforms
- Analytic Signals and Hilbert Transform Filters (http://ccrma-www.stanford.edu/~jos/r320/Analytic_Signals_Hilbert_Transform.html)
- Hilbert Transform (http://www.maths.utas.edu.au/People/McLean/tex/temp/node9.html) — Contains small table of transforms

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Hilbert Transform

The Hilbert Transform of the signal $x(t)$ is defined to be the signal whose frequency components are all phase shifted by $-\frac{\pi}{2}$ radians. The resulting signal is denoted

$$\hat{x}(t) = \mathcal{H}\{x(t)\}. \qquad (2.38)$$

$\hat{x}(t)$ is produced by passing $x(t)$ through a filter with transfer function

$$H(f) = -j\mathrm{sgn}(f) \qquad (2.39)$$

The magnitude and phase of $H(f)$ are

$$\begin{aligned} |H(f)| &= 1 && (2.40) \\ \angle H(f) &= -\frac{\pi}{2}\mathrm{sgn}(f) && (2.41) \end{aligned}$$

The impulse response is the inverse Fourier transform of (gif):

$$h(t) = \frac{1}{\pi t}. \qquad (2.42)$$

It is instructive to contrast and compare the transfer function of the Hilbert transmform to that of a pure time delay ($x(t) \rightarrow x(t - t_0)$). The transfer function of the time delay is

$$\begin{aligned} H_d(f) &= e^{-2\pi f t_0} && (2.43) \\ |H_d(f)| &= 1 && (2.44) \\ \angle H(f) &= -2\pi f t_0 && (2.45) \end{aligned}$$

Both have the same magnitude but the time delay has a phase which is linear in frequency instead of constant.

Example 1: Find $\hat{x}(t)$ when $x(t) = A\cos 2\pi f_0 t$.

$$
\begin{aligned}
\hat{x}(t) &= A\cos(2\pi f_0 t - \pi/2) \\
&= A\sin 2\pi f_0 t && (2.46) \\
\hat{X}(f) &= -j\mathrm{sgn}(f)\cdot\frac{A}{2}[\delta(f+f_0)+\delta(f-f_0)] \\
&= \frac{A}{j2}[-\delta(f+f_0)+\delta(f-f_0)] && (2.47) \\
\hat{x}(t) &= A\sin 2\pi f_0 t && (2.48)
\end{aligned}
$$

Example 2: Homework problem.

The properties of the Hilbert transform are outlined in Section 2.9 of the Text.

(Note: the Hilbert transform is used in complex analysis to generate complex-valued analytic functions from real functions. A function is analytic if and only if its components are harmonic conjugates. The Hilbert transform is used to generate a functions whose components are harmonic conjugates.)

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Sat Jan 4 14:45:02 MST 1997